# Approximation of small amplitude atmospheric waves short in vertical

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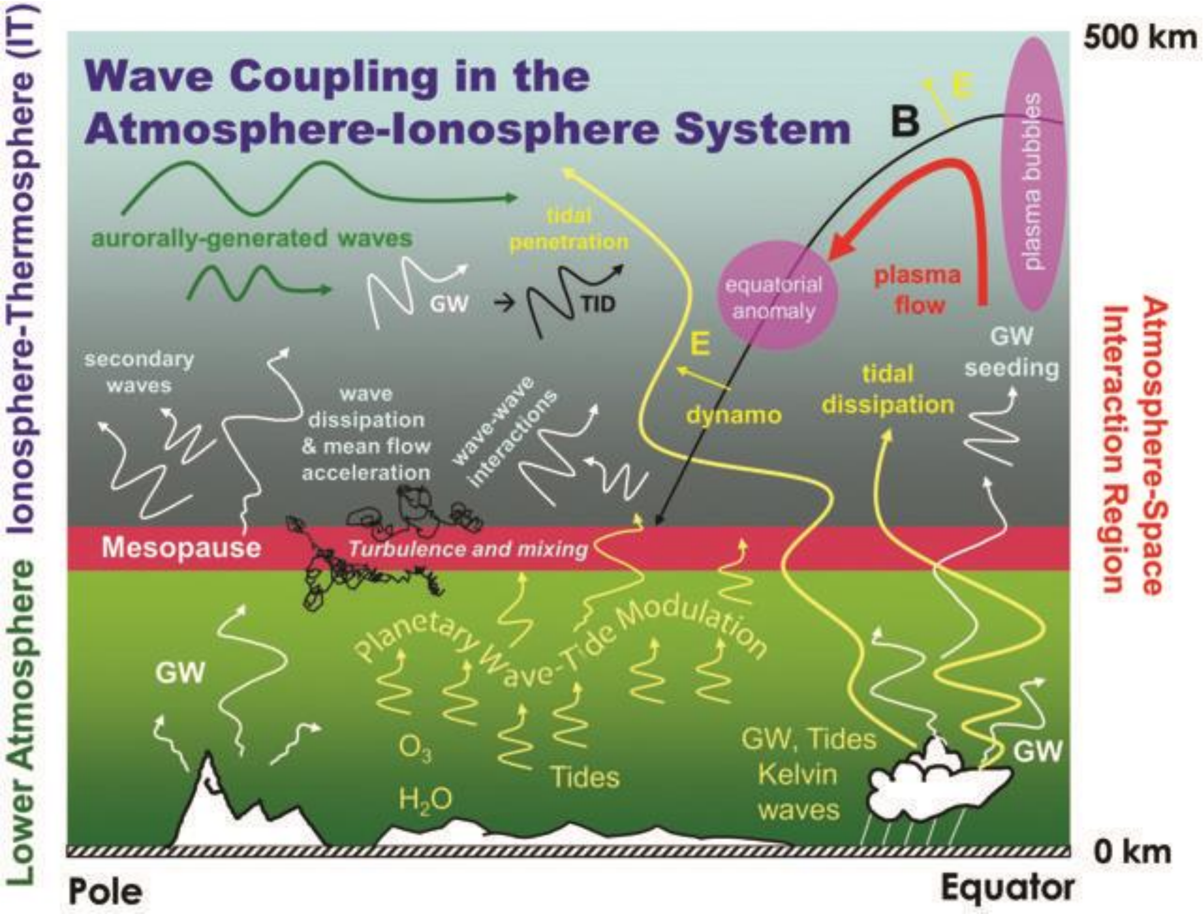
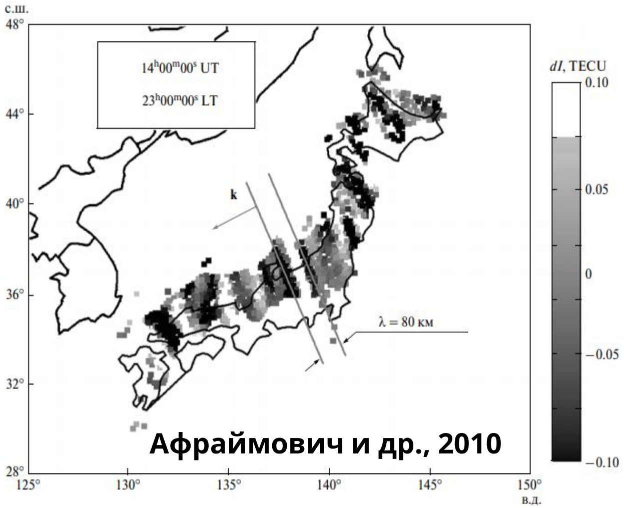
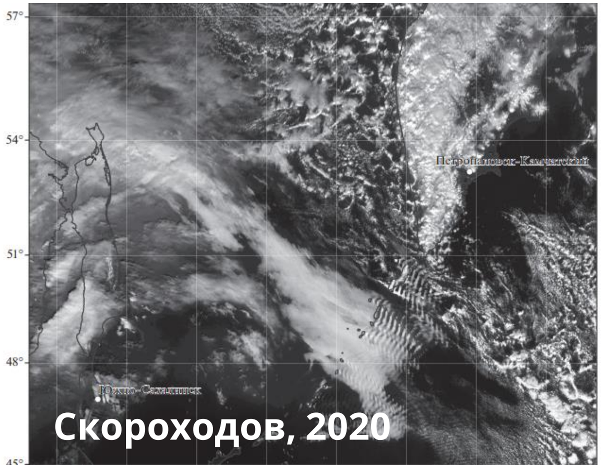
# Atmospheric waves

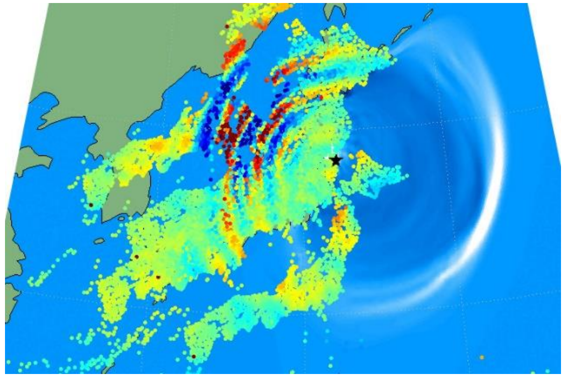


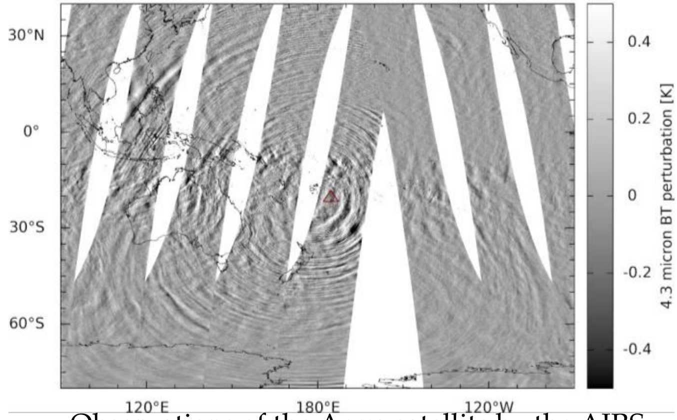

Structure of total electron content variations after passing through the solar terminator




Cloud "print" of internal gravity waves over the northern islands of the Kuril Archipelago



The Tohoku-Oki earthquake and the observed disturbances in the upper layers of the Earth's atmosphere




Observations of the Aqua satellite by the AIRS atmospheric infrared probe during and after the eruption of the Tongo volcano

# Points

- Using the method of multiple scales, formulas for the hydrodynamic fields of acoustic-gravity waves (AGWs) with vertical wavelengths small compared to the scales of changes in the background temperature and wind fields are derived. These formulas are equivalent to the traditional WKB approximation, but explicitly include the vertical gradients of the background fields.

- The conditions for the applicability of the obtained formulas to describe the propagation of AGWs from the troposphere to the thermosphere are formulated and analyzed.

- The search for singular points (critical levels) in approximate differential equations for wave modes is discussed. The issue of turning points and their dependence on AGW parameters and atmospheric parameters is analyzed.

# EQUATIONS FOR AGWs OF SMALL AMPLITUDE IN THE ATMOSPHERE WITH ACCOUNT FOR THE BACKGROUND WIND

Let us consider a system of three-dimensional linearized hydrodynamic equations for a non-dissipative atmospheric gas, without taking into account the rotation and curvature of the Earth's surface. The system of equations has the following form:

$$\frac{\partial\psi}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\alpha(z)+1}{\gamma H(z)}w+\frac{\partial w}{\partial z}+\frac{\partial\psi}{\partial x}U(z)+\frac{\partial\psi}{\partial y}V(z)=0,$$

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}U(z)+\frac{\partial u}{\partial y}V(z)+w\frac{d}{dz}U(z)+gH(z)\left(\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial x}\right)=0,$$

$$\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}U(z)+\frac{\partial v}{\partial y}V(z)+w\frac{d}{dz}V(z)+gH(z)\left(\frac{\partial\psi}{\partial y}+\frac{\partial\varphi}{\partial y}\right)=0,$$

$$\frac{\partial w}{\partial t}+\frac{\partial w}{\partial x}U(z)+\frac{\partial w}{\partial y}V(z)-\frac{\alpha(z)+1}{\gamma}g(\psi+\varphi)+g\frac{d}{dz}$$

$$\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial x}U(z)+\frac{\partial\varphi}{\partial y}V(z)+\frac{(\alpha(z)-\gamma+1)w}{\gamma H(z)}+(\gamma-1)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0.$$

Here $u$ and $v$ – components of the horizontal velocity in the wave along the horizontal axes $x$ and $y$, $w$ – velocity along the vertical $z$-axis. $U(z)$, $V(z)$– average velocity components. $\psi=\frac{\rho-\rho_0(z)}{\rho_0(z)}$ , $\varphi=\frac{T-T_0(z)}{T_0(z)}$ , were $\rho$ and $T$ – density and temperature, $\rho_0(z)$ and $T_0(z)$ – background density and temperature. The dependence of the background density on z is described by the formula $\rho_0(z)=\frac{\rho_{00}}{H(z)}exp\left(-\int_0^z\frac{dz}{H(z)}\right)$, and $\frac{d\rho_0(z)}{dz}=\frac{-\alpha(z)+1}{\gamma H(z)}\rho_0(z)$ and $H(z)=\frac{RT_0(z)}{g\mu}$– homogeneous atmosphere height. $R$ – universal gas constant, $\mu$ – molecular weight, $\gamma$ – adiabatic constant, $g$ – Gravitational acceleration. Derivatives of $\mu,\gamma,g$ can be neglected. Variable $\alpha(z)=\gamma-1+\gamma\frac{dH(z)}{dz}>0$.

# Equations and Relations for Wave Modes

For wave modes $\sim exp\left(i(kx + my - \omega t)\right)$ this system of equations is reduced to the i-form

$$\frac{d}{dz}w(z) + \left(\frac{-i\sigma(z)}{gH(z)\gamma} + \frac{-i\left(m^2+k^2\right)}{\sigma(z)}\right)\frac{p(z)}{\rho_0(z)} - \left(\frac{1}{\gamma H(z)} - \frac{k\frac{d}{dz}U(z)+m\frac{d}{dz}V(z)}{\sigma(z)}\right)w(z) = 0,$$

$$\frac{d}{dz}p(z) + \frac{p(z)}{\gamma H(z)} + \left(-i\sigma(z) + \frac{-ig\alpha(z)}{\gamma H(z)\sigma(z)}\right)\rho_0(z)w(z) = 0,$$

where functions are introduced $p(z) = \rho_0(z)gH(z)(\phi + \psi)$, and $\sigma(z) = \omega - kU(z) - mV(z)$.

## Approximation of vertical short waves

In this study, method of multiple scales will be used. Within the framework of this approach, we consider that the function $H, U, V, \alpha$, included in the coefficients of the equations are slowly changing functions of the variable $z$ that is, they depend on $z$ through the variable $\xi = \epsilon z$, and have the form $H(\xi), U(\xi), V(\xi), \alpha(\xi)$. Then in the equation above before the terms containing $\frac{d}{d\xi}U(\xi)$ and $\frac{d}{d\xi}V(\xi)$ contains a small parameter $\epsilon$.

In the course of constructing the perturbation theory, a local dispersion relation is derived that relates the parameters $m, k, \omega$ and the local vertical wave number $n(z)$ of the wave mode. Since the coefficients of the equations are described by slowly varying functions of the vertical coordinate, it is natural to look for an approximate solution of the equations above in the form

$$W(z) = A_w(\xi) * e^{S(z)}, p(z) = \rho_0(z) * A_p(\xi) * e^{S(z)}.$$

# Approximation of vertical short waves

Substituting the approximate solution into the system of equations, we obtain the relations

$$\left(\frac{-1}{H(\xi)\gamma}+\frac{d}{dz}S(z)+\frac{\left(m\frac{d}{d\xi}V(\xi)+k\frac{d}{d\xi}U(\xi)\right)i\epsilon}{\sigma(\xi)}\right)A_w(\xi)+i\left(\frac{-\sigma(\xi)}{gH(\xi)\gamma}+\frac{k^2+m^2}{\sigma(\xi)}\right)A_p(\xi)+\epsilon\frac{d}{d\xi}A_w(\xi)=0,$$

$$i\left(-\sigma(\xi)+\frac{g\alpha(\xi)}{gH\sigma(\xi)\gamma}\right)A_w(\xi)+\left(\frac{d}{dz}S(z)-\frac{\alpha(\xi)}{\gamma H(\xi)}\right)A_p(\xi)+\epsilon\frac{d}{d\xi}A_p(\xi)=0$$

Neglecting small terms $\epsilon\frac{d}{d\xi}A_w(\xi), \epsilon\frac{d}{d\xi}A_p(\xi)$ order $\epsilon$, we arrive at a system of linear homogeneous algebraic equations with respect to the unknowns $A_w$, $A_p$. The condition for the solvability of a system of linear homogeneous algebraic variables for the unknowns $A_w$, $A_p$ is the equality of the determinant of this system to zero:

$$\sigma^4(\xi)-\left((k^2+n^2(\xi)+m^2)gH(\xi)\gamma+\frac{(\alpha(\xi)+1)^2g}{4\gamma H(\xi)}\right)\sigma^2(\xi)-\left(-(\xi)\gamma H(\xi)+\frac{\alpha(\xi)-1}{2}\right)\left(k\frac{d}{d\xi}U(\xi)+m\frac{d}{d\xi}V(\xi)\right)g\epsilon\sigma(\xi)$$

$$-\left(-(\xi)\gamma H(\xi)+\frac{\alpha(\xi)-1}{2}\right)\left(k\frac{d}{d\xi}U(\xi)+m\frac{d}{d\xi}V(\xi)\right)g\epsilon\sigma(\xi)+\alpha(\xi)(k^2+m^2)g^2=0.$$

This relation has the meaning of the local dispersion relation. Here $n(\xi)$ has the meaning of the local vertical component of the wave vector.

# The dependence of the complex local vertical component $n(\xi)$

The dependence of the complex local vertical component $n(\xi)$ of the wave vector on от $k, m$ and frequency is written below

$$n(\xi) = \epsilon \frac{i\left(k\frac{d}{d\xi}U(\xi) + m\frac{d}{d\xi}V(\xi)\right)}{2\sigma(\xi)} \pm$$

$$\pm \frac{1}{2\gamma g H(\xi)\sigma(\xi)}\left[-2(\alpha(\xi)-1)\left(k\frac{d}{d\xi}U(\xi) + m\frac{d}{d\xi}V(\xi)\right)\gamma\sigma(\xi)g^2H(\xi)\epsilon - 4(k^2+m^2)\gamma^2\sigma^2(\xi)g^2H^2(\xi)\right.$$

# Approximation formulas for short waves along the vertical coordinate

We substitute the resulting expression into the first equation of the system of equations with an approximate ratio. As a result of some simplifications, omitting small orders of $A_p(\xi)$:

$$\left(\frac{\left(k\frac{d}{d\xi}U(\xi)+m\frac{d}{d\xi}V(\xi)\right)\epsilon}{\sigma(\xi)}+2\in(\xi)\right)\frac{dA_p(\xi)}{d\xi}+D(\xi)A_p(\xi)=0,$$

Let us represent the complex vertical component of the wave vector in the form $n(\xi)=n_1(\xi)+n_2(\xi)$, were $n_1(\xi)=\epsilon\frac{i\left(k\frac{d}{d\xi}U(\xi)+m\frac{d}{d\xi}V(\xi)\right)}{2\sigma(\xi)}$. With real $k,m,\omega$, when the expression in square brackets is positive, the function $n_2(\xi)$ is real, and therefore the role of the vertical component of the wave vector is played by $n_2(\xi)$.

The solution of the equation has the form

$$A_p(\xi)=exp\left(-\int_0^{\xi}\frac{D(\xi')}{2\in(\xi')}d\xi'\right).$$

The approximation formulas for short waves along the vertical coordinate are completely constructed. Note that there is some arbitrariness in the shortwave formulas for $A_p(\xi)$. For the wave short-wave mode, the wave parameters $k,m,\omega,n_2(\xi)$ are related by the obtained dispersion relation. Therefore, some of these parameters can be expressed in terms of others. In this case, the form of the formulas will change, although the formulas will be equivalent.

# ANALYSIS OF THE CONDITIONS OF APPLICABILITY OF WKB FORMULAS

Singular points

- According to the mathematical theory of the short-wave approximation, the written formulas are valid if all the coefficients of the equations are described by doubly differentiable functions, and if the waves are sufficiently short along the vertical coordinate $z$ : $\epsilon \ll 1$.
- Some coefficients of the equations

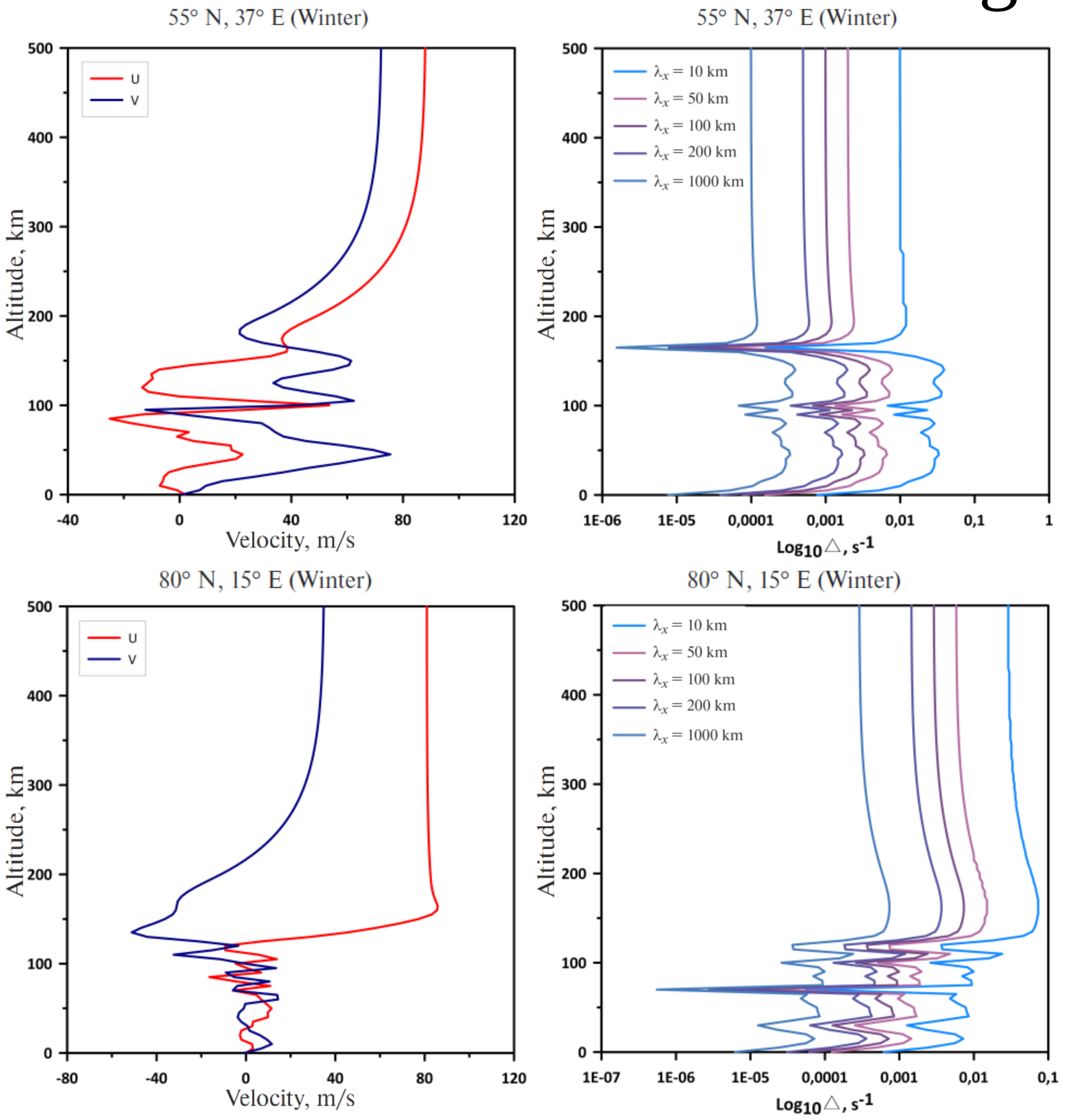
$$\frac{d}{dz}w(z) + \left(\frac{-i\sigma(z)}{gH(z)\gamma} + \frac{-i\left(m^2+k^2\right)}{\sigma(z)}\right)\frac{p(z)}{\rho_0(z)} - \left(\frac{1}{\gamma H(z)} - \frac{k\frac{d}{dz}U(z)+m\frac{d}{dz}V(z)}{\sigma(z)}\right)w(z) = 0,$$

- $$\frac{d}{dz}p(z) + \frac{p(z)}{\gamma H(z)} + \left(-i\sigma(z) + \frac{-ig\alpha(z)}{\gamma H(z)\sigma(z)}\right)\rho_0(z)w(z) = 0$$

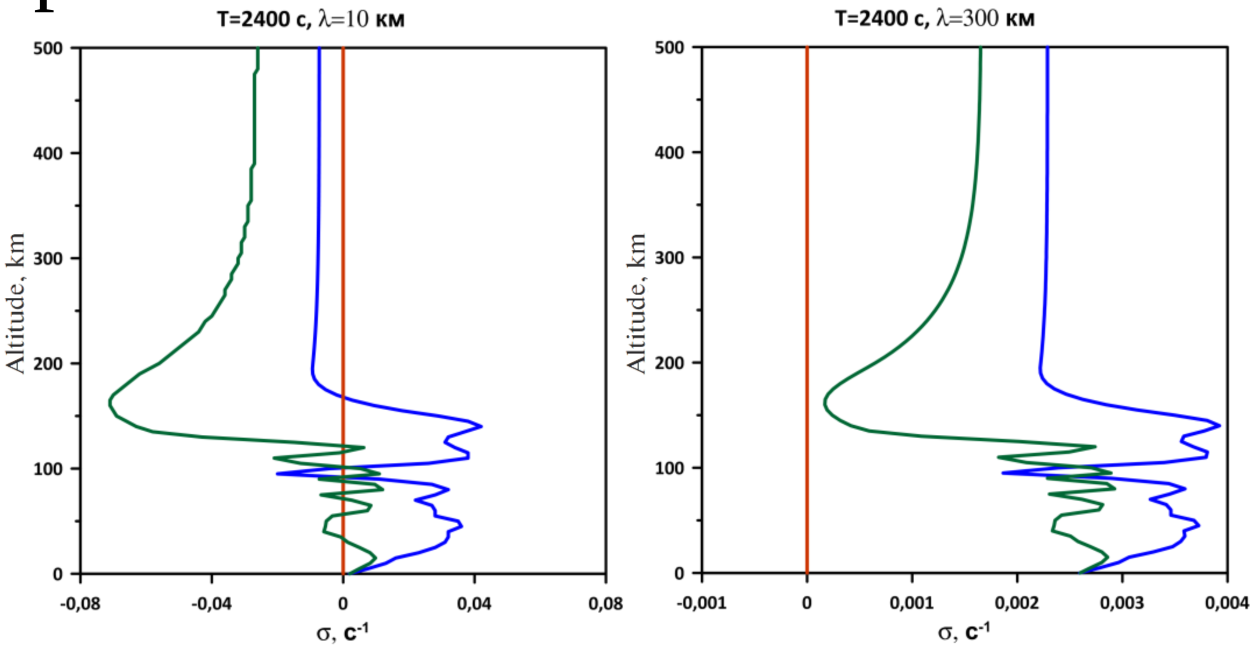
contain a function in the denominator $\sigma(z)=\omega - kU(z) - mV(z)$, which can take a value equal to zero at some heights. In this case, the corresponding coefficients turn to infinity. These equations may contain singular points at heights $z_0$, where $\sigma(z)=0$.

- If a singular point in the equations is present in the altitude interval from the troposphere to the upper atmosphere, then the conditions for the applicability of the short-wave approximation are not met in this altitude interval, and the short-wave formulas may not be accurate enough to describe the propagation of waves from tropospheric altitudes to the thermosphere.

# Singular point




The left panel shows the zonal and meridional wind components of the HWM empirical horizontal wind model. Right panel shows Doppler frequency Shift at λ=10, 50, 100, 200, 1000 km.




Schedule $\sigma(z)$ for waves with frequency $\omega = 2\pi/2400s$, wavelength λ =10 km (left panel), and λ = 300 km (right panel). The red line represents zero.

The presence of singular points is typical for shorter waves, with a horizontal scale of less than or about 10 km. The number of singular points decreases with increasing wavelength. For waves with a horizontal scale of the order of or greater than 300 km, there are usually no singular points.

# ANALYSIS OF THE CONDITIONS OF APPLICABILITY OF WKB FORMULAS

Turning points

From the derived formulas it is clear that when $n_2(z) = 0$, then the denominator of the integrand goes to zero, and the integrand itself goes to infinity. Thus, the assumption used in deriving the formulas of the WKB approximation is that $A_p(\xi)$, $A_w(\xi)$ are slowly varying functions of the variable z is obviously violated in the vicinity of the point, where $n_2(z) = 0$.

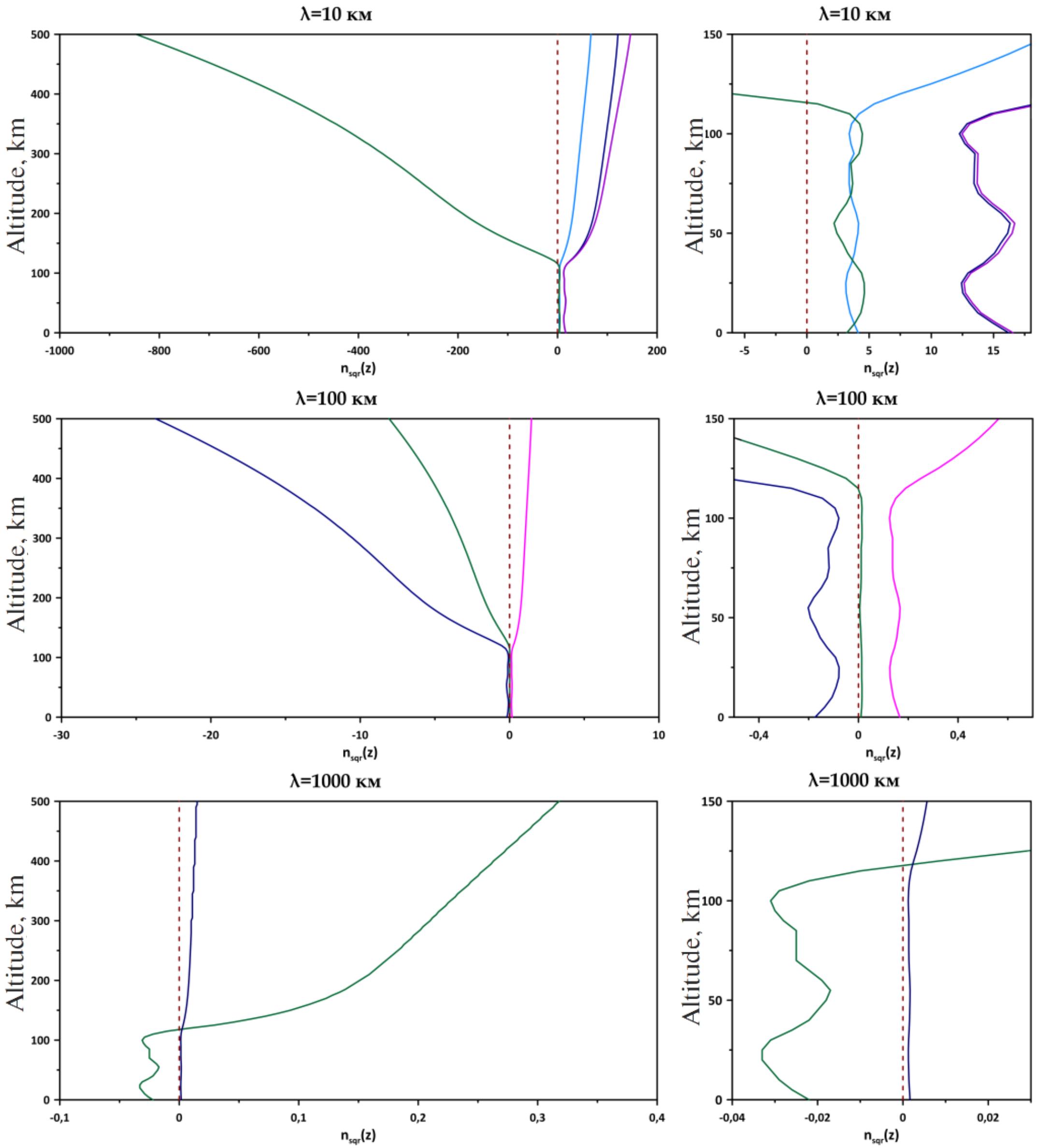
Points, when $n_2(z) = 0$, are called turning points. Turning points are usually interpreted physically as levels from which the wave is reflected, because beyond the turning point the wave attenuates. To fulfill the condition $n_2(z) = 0$ it is enough that the radical part of the relation for complex local vertical component $n(\xi)$ was equal to zero.

$$n(\xi) = \epsilon \frac{i\left(k\frac{d}{d\xi}U(\xi) + m\frac{d}{d\xi}V(\xi)\right)}{2\sigma(\xi)} \pm$$

$$\pm \frac{1}{2\gamma g H(\xi)\sigma(\xi)}\left[-2(\alpha(\xi) - 1)\left(k\frac{d}{d\xi}U(\xi) + m\frac{d}{d\xi}V(\xi)\right)\gamma\sigma(\xi)g^2H(\xi)\epsilon - 4(k^2 + m^2)\gamma^2\sigma^2(\xi)g^2H^2(\xi)\right.$$

## Turning points

It can be seen that, at low frequencies on small and medium horizontal scales of the wave, graph $n_{sqr}(z)$ always has one point $n_{sqr}(z) = 0$ at altitudes from 110 to 130 km. The turning points coming from the troposphere of the wave from the specified range are reflected down at these heights. At low frequencies with large wave scales, there is another turning point; it indicates that waves from the specified range propagating in the upper atmosphere do not penetrate to tropospheric heights.




The left panel is the graph of the function $n_{sqr}(z)$ for values $\omega$ = 6.3x10$^{-6}$, 3.0x10$^{-3}$, 1.9x10$^{-2}$ s$^{-1}$ (green, blue and pink line, respectively) and $\omega = \sqrt{\frac{q(\gamma-1)}{2H}}$ (light blue) at $\lambda_x$ = 10, 100, 1000 км. The red dashed line is zero $n_{sqr}(z)$. The right panel shows graphs of the left panel enlarged in the zero area.

# Conclusions

- Using the method of different scales, formulas for vertical profiles of wave modes in the approximation of waves short in the vertical variable are derived and analyzed, convenient for parameterizing internal gravity waves. The derived formulas are complete and more subtly take into account the change in atmospheric parameters with height, in comparison with other versions of the approximation of short waves along the vertical.
- Applicability conditions for approximate formulas for describing AGWs propagating from tropospheric heights to thermospheric heights are formulated. One condition for the applicability of formulas is the absence of singular points at which $\sigma(z) = 0$ in the range of heights from tropospheric to thermospheric. For typical background wind profiles, there are no singular points for acoustic waves. For IGWs, there are usually singular points in the equations, and there may be several of them at the same time.
- The presence of singular points is characteristic of shorter waves, with a horizontal scale of less than or on the order of 10 km. The number of singular points decreases with increasing wavelength. For waves with a horizontal scale of about 300 km or more, there are usually no singular points. It is shown that the discussed singular points are simple for which the solutions in the vicinity of the singular point are mathematically investigated.
- Another condition for the applicability of the shortwave approximation formulas is the absence of turning points in the height interval from the troposphere to the upper atmosphere.

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# Thank you for your attention!