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Chaotic regimes in hereditary two-mode dynamo models

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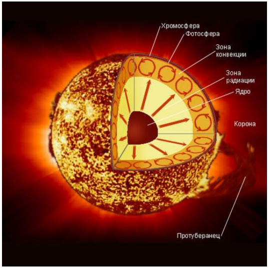
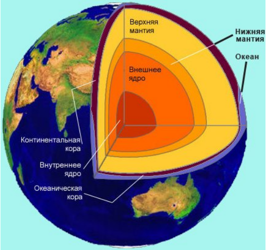
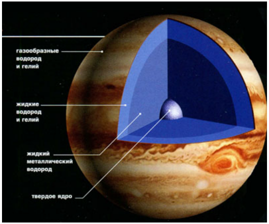
Relevance

If an electrically conducting environment (liquid, gas) moves in a weak external magnetic field, then with a certain geometry of this movement a new magnetic field can be generated in the environment, which is much greater than the original one. This field, in turn, changes the nature of the motion of the medium. A self-consistent dynamo system emerges..

This mechanism explains the existence of large-scale magnetic fields of space objects - planets and their satellites, stars, galaxies.







Initial Equations

The magnetic field induction equation in the Ω region filled with a conducting environment has the form

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{V} \times \mathbf{B}) + \nu_m \Delta \mathbf{B} \\ \nabla \mathbf{B} &= 0 \end{aligned} \quad (1)$$

where $\mathbf{V}$ is the velocity field of the environment, and ν_m is the magnetic viscosity, assumed by us to be constant.
Mean field theory introduces expansion of the velocity and magnetic fields into mean fields $\mathbf{U}$ and $\mathbf{B}$ and fluctuations $\mathbf{u}$ and $\mathbf{b}$.

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times (\alpha \mathbf{B}) + \beta \Delta \mathbf{B} \\ \nabla \mathbf{B} &= 0 \end{aligned} \quad (2)$$

Here α is the α-effect tensor defined by the equality $\alpha B = < u \times b >$.

Initial Equations

The dimensionless form of the equation (2) with the preservation of the designation of the fields will look like:

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times (\widetilde{\alpha} \mathbf{B}) + {R_m}^{-1} \Delta \mathbf{B} \\ \nabla \mathbf{B} &= 0 \end{aligned} \tag{3}$$

where R_m is the magnetic Reynolds number, $\widetilde{\alpha}$ is the dimensionless α-effect tensor.
For the axisymmetric case under consideration, the equation (3) is split into equations (4) for the toroidal and poloidal components of the magnetic field.

$$\begin{aligned} \frac{\partial \mathbf{B}^T}{\partial t} &= \mathrm{rot}(\mathbf{U} \times \mathbf{B}^P) + \mathrm{rot}(\widetilde{\alpha} \mathbf{B}^P) + {R_m}^{-1} \Delta \mathbf{B}^T \\ \frac{\partial \mathbf{B}^P}{\partial t} &= \mathrm{rot}(\widetilde{\alpha} \mathbf{B}^T) + {R_m}^{-1} \Delta \mathbf{B}^P \end{aligned} \tag{4}$$

Two-mode approximation

The spatial structure of the magnetic field is assumed to be very simple and is represented by one poloidal and one toroidal mode:

$$\mathbf{B} = \mathbf{B}^T + \mathbf{B}^P = B^T(t)\mathbf{b}^T(\mathbf{r}) + B^P(t)\mathbf{b}^P(\mathbf{r}) \tag{5}$$

We substitute the expansion (5) into the induction equation for the modes (4) and by the Galerkin method we obtain a dynamical system for the amplitudes $B^T(t)$ and $B^P(t)$:

$$\begin{aligned} \frac{dB^T}{\partial t} &= \omega B^P + \alpha^T B^P - \eta^T B^T \\ \frac{dB^P}{\partial t} &= \alpha^P B^T - \eta^P B^P \end{aligned} \tag{6}$$

System coefficients

$$
\begin{aligned}
\omega &= \frac{1}{\|\mathbf{b}^T\|^2} \int_\Omega \left[\nabla \times (\mathbf{U} \times \mathbf{b}^P)\right] \mathbf{b}^T dr > 0 \\
\eta^T &= \frac{-R_m{}^{-1}}{\|\mathbf{b}^T\|^2} \int_\Omega (\Delta \mathbf{b}^T)\mathbf{b}^T dr > 0 \\
\eta^P &= \frac{-R_m{}^{-1}}{\|\mathbf{b}^P\|^2} \int_\Omega (\Delta \mathbf{b}^P)\mathbf{b}^P dr > 0 \\
\alpha^T &= \frac{1}{\|\mathbf{b}^P\|^2} \int_\Omega \left(\nabla \times \widetilde{\alpha}\mathbf{b}^P\right) \mathbf{b}^T dr > 0 \\
\alpha^P &= \frac{1}{\|\mathbf{b}^T\|^2} \int_\Omega \left(\nabla \times \widetilde{\alpha}\mathbf{b}^T\right) \mathbf{b}^P dr > 0 \\
\|\mathbf{b}^T\|^2 &= \int_\Omega \left[\mathbf{b}^T(r)\right]^2 dr \\
\|\mathbf{b}^P\|^2 &= \int_\Omega \left[\mathbf{b}^P(r)\right]^2 dr
\end{aligned} \tag{7}
$$

Axisymmetric $\alpha^2\omega$-dynamo model

We introduce a dynamic correction into the intensities α^T, α^P to provide suppression and take the representation $\alpha^P = \alpha_0 - w$, $\alpha^T = \xi(\alpha_0 - w)$, where ξ is a dimensionless coefficient.
We get a system of the form:

$$\begin{aligned} \frac{dB^T}{\partial t} &= (\omega + \xi(\alpha_0 - w))B^P - \eta^T B^T \\ \frac{dB^P}{\partial t} &= (\alpha_0 - w)B^T - \eta^P B^P \\ w &= Q(B^T, B^P) \end{aligned} \tag{8}$$

In the system (8), we introduce the change of variables:

$$x_1(t) = B^T(t), \quad x_2(t) = sB^P(t), \quad x_3(t) = \frac{w}{s},$$

where: $s = (\omega + \xi\alpha)/\eta^T$ and $D = s\alpha/\eta^P$. We also choose the characteristic dissipation time of the poloidal component of the magnetic field as the time scale, then $\eta^P = 1$.

α-effect suppression model

The dependence $Q\left(B^T, B^P\right)$ must be quadratic in the field components from physical considerations.
Algebraic suppression (explored before):

$$\alpha = \alpha_0 - Q(B^T(t), B^P(t)), \tag{9}$$

where Q is a quadratic function.

Dynamic suppression (explored previously):

$$\alpha = \alpha_0 - x_3(t), \tag{10}$$

where: $Dx_3 = Q(B^T(t), B^P(t))$ and D is a differential operator.

Hereditary suppression (first proposed in work):

$$\alpha = \alpha_0 - x_3(t), \tag{11}$$

where:

$$x_3(t) = \int_0^t K(t-\tau)Q(x_1(\tau), x_2(\tau))\, d\tau$$

The kernel of the suppression functional $K(t)$ is a rather arbitrary function with the following properties: $K(t) \geq 0 \quad \forall t \geq 0$ and $K(+\infty) = 0$.

Fixing the kernel K and the quadratic form Q determines a specific suppression model.

Representation in the form of the Volterra equation

The model equation can be written in the form of the Volterra vector equation:

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathbf{K}(t-\tau)\mathbf{f}(\mathbf{x}(\tau))d\tau \qquad (12)$$

where

$$\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T \qquad \mathbf{x}(0) = [x_1(0), x_2(0), 0]^T$$

$$\mathbf{K}(t-\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K(t-\tau) \end{bmatrix} \mathbf{f}(\mathbf{x}(\tau)) = \begin{bmatrix} \left(\eta^T - \frac{\xi}{s^2} x_3(\tau)\right) x_2(\tau) - \eta^T x_1(\tau), \\ (D - x_3(\tau)) x_1(\tau) - x_2(\tau) \\ Q(x_1(\tau), x_2(\tau)) \end{bmatrix}$$

A theorem on the existence and uniqueness of a solution using the fixed point principle was proved in [Kazakov,2022].

On the possibility of eliminating the integral term

For some classes of kernels, the integro-differential equations of the model are equivalent to differential equations with an increase in the dimension of the phase space of a dynamical system.

Theorem: If the core $K(t)$ - differential equation solution

$$a_0 K^{(n)}(t) + a_1 K^{(n-1)}(t) + \cdots + a_{n-1} K'(t) + a_n K(t) = 0, \quad a_i - \text{const}, \tag{13}$$

then the integral representation

$$x_3(t) = \int_0^t K(t-\tau) Q(x_1(\tau), x_2(\tau))\, d\tau$$

is equivalent to a differential equation of order n:

$$a_0 \frac{d^n x_3}{dt^n} + a_1 \frac{d^{n-1} x_3}{dt^{n-1}} + \ldots a_n x_3 = \sum_{k=0}^{n} a_{n-k} \sum_{m=0}^{k-1} K^{(m)}(0) \frac{d^{k-m-1}}{dt^{k-m-1}} Q(x_1(t), x_2(t))$$

for the function $x_3(t)$.

All such kernels have exponential asymptotics, i.e., for them we can talk about the effective memory duration. If the kernel is not integrable on numerical direct memory and is infinitely long, reduction to classical differential systems is impossible.

Dynamic mode maps constructed using Lyapunov exponents

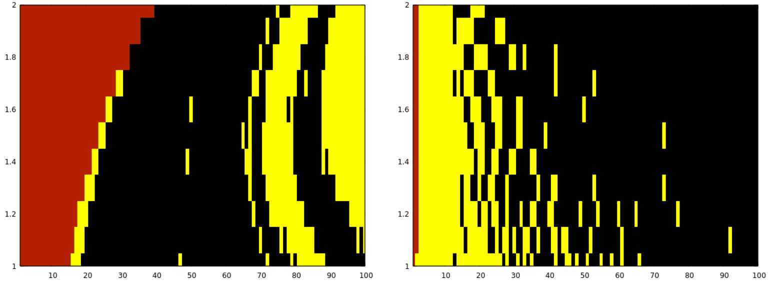


Рис.: Map of dynamic regimes for $\alpha\omega$-dynamo systems with kernel (a) $K(t) = e^{-bt}$; (b) $K(t) = te^{-bt}$.

Algorithm for constructing maps of dynamic modes

Autocorrelation function

$$R(\tau) = \sum_{t=0}^{N} x(t)x(t+\tau) \tag{14}$$

Discrete signal energy

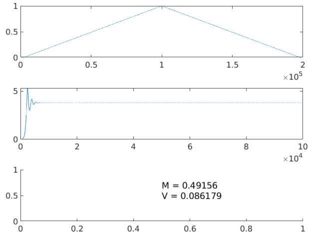
$$E = \sum_{t=0}^{N} |x^2(t)| \tag{15}$$

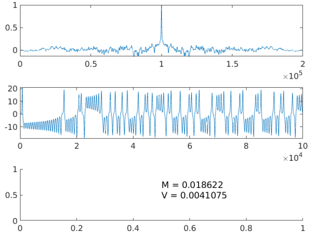
We normalize the series $R(\tau)$ to the signal energy

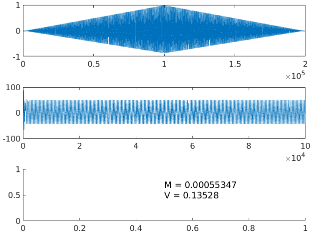
$$L(\tau) = \frac{R(\tau)}{E} \tag{16}$$

Algorithm for constructing maps of dynamic modes

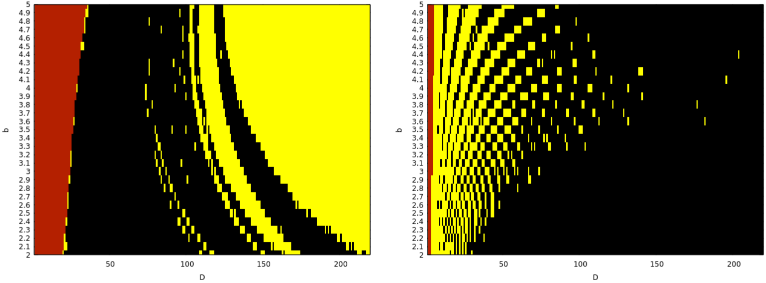
M	V	Type
>=0.4	-	Stationary mode
<0.4	<0.01	Chaotic mode
<0.4	>0.01	Periodic mode







Dynamic mode maps



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Dynamic mode maps

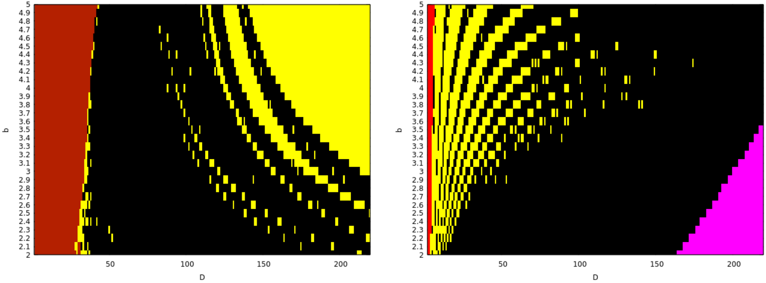

Рис.: Map of dynamic regimes for $\alpha\omega$-dynamo systems with kernel (a) $K(t) = \frac{1}{(t+1)^{\alpha}}$; (b) $K(t) = \frac{t}{(t+1)^{\alpha+1}}$.

Summary

1. An hereditary model of a two-mode dynamo is proposed, covering the cases of axisymmetric large-scale α^2-, $\alpha\omega$- and $\alpha^2\omega$-dynamo.
2. A theorem is proved on the possibility of eliminating the hereditary term for the class of kernels with exponential asymptotics, i.e. about the finiteness of memory in this case.
3. A theorem on the existence and uniqueness of solutions for systems of this type is proved.
4. The model reproduces the dynamic regimes observed in real space dynamo systems.
5. The constructed maps of dynamic regimes will help in the further study of the properties of hereditary models of a two-mode dynamo.

Thank you for your attention!