The power-law compound and fractional Poisson process in the theory of anomalous phenomena

Степенной составной и дробный процесс Пуассона в теории аномальных явлений

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Цель(Purpose):

The power-law compound and time-fractional Poisson process is considered as a statistical model of anomalous phenomena in the ereditarian theory of criticality. This model can be useful in studies of energy-active zones. Regardless of their nature, anomalous phenomena have universal statistical properties, among which, first of all, it should be noted scale invariance. In the proposed model, the special role of scaling in the properties of anomalous phenomena is shown taking into account the eredite effects, the physical meaning of which is explained by examples of analogies between anomalous phenomena of different nature. Critical process modes and exceptional values of critical indexes are determined. The structural instability of the process caused by scaling and catastrophes in its statistical characteristics are discussed. The obtained results are used to study seismic data and determine the critical indices of the deformation process.

Степенной составной и дробный по времени процесс Пуассона рассматривается как статистическая модель аномальных явлений в эредитарной теории критичности. В предложенной модели особая роль скейлинга (масштабной инвариантности) в свойствах аномальных явлений показана с учетом эредитарных эффектов, физический смысл которых объясняется на примерах аналогий между аномальными явлениями различной природы. Определены критические режимы процесса и исключительные значения критических индексов. Обсуждается обусловленная скейлингом структурная неустойчивость процесса и катастрофы в его статистических характеристиках. Полученные результаты используются для исследования сейсмических данных и определения критических индексов деформационного процесса.

Compound fractional Poisson process

Equations of Compound Fractional Poisson Process (CFPP) of order k with integer random state changes by $r = 1,\ 2, ...,\ k$ can be represented as [1]:

$$\begin{cases} \dfrac{dp_0^\nu(t)}{dt^\nu} = -\Lambda p_0^\nu(t), & \Lambda = \sum\limits_{r=1}^{k} \lambda_r, \\ \dfrac{dp_j^\nu(t)}{dt^\nu} = \sum\limits_{r=1}^{j} \lambda_r p_{j-r}^\nu(t) - \Lambda p_j^\nu(t), & j = 1,\ 2, ..., k-1, \\ \dfrac{dp_j^\nu(t)}{dt^\nu} = \sum\limits_{r=1}^{k} \lambda_r p_{j-r}^\nu(t) - \Lambda p_j^\nu(t), & j = k,\ k+1, ..., \end{cases} \tag{1}$$

with initial conditions

$$p_j^\nu(0) = \begin{cases} 1, & j = 0, \\ 0, & j \geq 1, \end{cases}$$

where ν is the exponent of the fractional derivative, $0 < \nu \leq 1$.

Series of repeatability frequencies of events

The power-law distribution of dislocation changes in the deformation process can be determined using the Gutenberg-Richter law, if it is written in discrete form for the frequency of occurrence of events:

$$\omega_r = 2b\Omega \cdot r^{-2b-1}, \ \sum_{r=1}^{k} \omega_r = \Omega = \frac{N_{total}}{T}, \tag{2}$$

where Ω is the total frequency of events, N_{total} is the total number of events, b is b-value.

Equations (1) are supplemented by a power-law distribution (2) of frequency ω_r of random events occurrence with jump amplitude $r = 1, 2, ..., k$, which are related to the parameters λ_r of equations (1) (fractional event frequencies) as follows:

$$\lambda_r = \omega_r^{\nu} = \left(2b\Omega \cdot r^{-2b-1}\right)^{\nu}, \tag{3}$$

$$\sum_{r=1}^{k} \lambda_r = \Lambda = (2b\Omega)^{\nu} \sum_{r=1}^{k} r^{-(2b+1)\nu}. \tag{4}$$

Determination of the parameter b of the distribution of repeatability frequencies ω_r

Distribution of the frequencies ω_r [day^{-1}] of repeatability of events

$$\omega_r = 2b\Omega \cdot r^{-2b-1}$$

is defined by the parameter b of the Gutenberg-Richter law (Fig.1).

We use to calculating the parameter b:

- the earthquake catalog of the Kamchatka Branch of the Geophysical Survey RAS for the Kuril-Kamchatka island arc subduction zone ($46^\circ - 62^\circ$ N, $158^\circ - 174^\circ$ E) [2],
- the period from 1 January 1962 to 31 December 2002,
- the earthquakes of energy classes $K \in [8.3,\ 16.1]$,
- the size $n = 46917$ earthquakes.

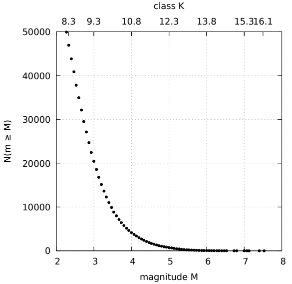

Рис. 1: The empirical Gutenberg-Richter law

We use to approximating linear part of the logarifmic empirical Gutenberg-Richter law (Fig.2) (finding of the approximation interval):

- the exponential approximated function of the empirical Gutenberg-Richter law,
- the least squares method,
- the approximation error $1\% < \varepsilon < 10\%$ and $\varepsilon \to Min$ when changing the length of the interval $K \in [8.3,\ 16.1]$ by deleting events from its beginning and end,
- the largest correlation index R,
- the approximation error of the logarithmic Gutenberg-Richter law $1\% \leq \varepsilon' \leq 2\%$ and $\varepsilon' \to Min$.

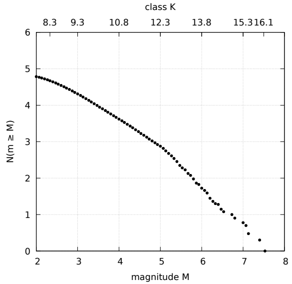

Рис. 2: The logarithmic empirical Gutenberg-Richter law

Таблица 1: Statistical characteristics of approximating functions

Function of approximation	Interval of approximation $[K_1,\ K_2]$	Interval of approximation $[M_1,\ M_2]$	n_K [1]	N_{total}	b	R	$\varepsilon, \%$
$Y = 10^{a-bX}$	$[9.2,\ 12.9]$	$[2.93,\ 5.40]$	38	22230	0.6897	0.986	1.658
$\lg Y = a - bX$						0.857	1.687

[1] n_K – number of classes

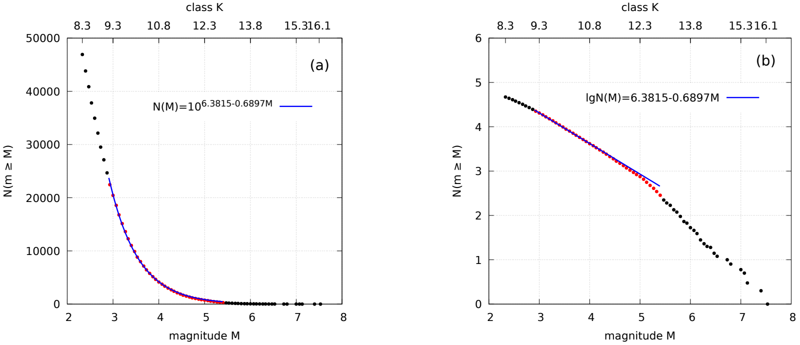

Рис. 3: The empirical Gutenberg-Richter law: (a) exponential, (b) logarithmic. The blue graph – nonlinear regression, the dot graph – empirical law, where the red dots – approximation interval.

Distributions of the first-passage times

The probability distributions of the waiting times for the first movement for each scale r are represented by expressions [1]

$$P(t) = \lambda_r t^{\nu_r} E_{\nu_r,\nu_r+1}(-\lambda_r t^{\nu_r}),\ \ t \geq 0,\ \ r = 1, 2, \ldots, k, \tag{5}$$

where $E_{\nu,\nu+1}(x)$ is the Mittag-Leffler function, $\lambda_r = \omega_r^{\nu_r}$ – fractional frequency of repeatability of events, $k = 38$.

Таблица 2: Parameters of first-passage times distributions

				Two-parameter approximation by a function (5)			
r	K_r	M_r	n_r[1]	RSS	ω_r, [day^{-1}]	ν_r	ε, %
1	2	3	4	5	6	7	8
1	9.2	2.93	57	0.025	0.182	0.891	2.33
19	11.0	4.13	113	0.041	0.033	0.865	3.76
26	11.7	4.6	95	0.023	0.021	0.784	4.07
38	12.9	5.4	50	0.032	0.006	0.883	5.32

$$\Lambda = \sum_{r=1}^{38} \lambda_r = \sum_{r=1}^{38} \omega_r^{\nu_r} = 1.8654\ day^{-0.7849} = (2.2129/day)^{0.7849},$$

where $\nu = 0.7849$ – the average exponent of the fractional derivative (Table 2, col.7).

[1] n_r – number of inter-event time intervals

The mean and variance

The mean $\mathbf{E}(t)$ and variance $\mathbf{Var}(t)$ of the CFPP [1]:

$$\mathbf{E}(t) = S_{k,1}(2b\Omega \cdot t)^{\nu}/\Gamma(\nu+1), \quad t \geq 0,$$
$$\mathbf{Var}(t) = S_{k,2}(2b\Omega \cdot t)^{\nu}/\Gamma(\nu+1) + \big(S_{k,1}(2b\Omega \cdot t)^{\nu}\big)^2 Z(\nu), \quad t \geq 0,$$
$$S_{k,p} = \sum_{r=1}^{k} r^{-(2b+1)\nu+p}, \; p = 1, 2, \tag{6}$$
$$Z(\nu) := \frac{1}{\nu}\left(\frac{1}{\Gamma(2\nu)} - \frac{1}{\nu\Gamma^2(\nu)}\right),$$

where $\Gamma(x)$ is the gamma function.

Results

As a result of two-parameter approximation of the distributions of the first-passage times (Table 2), the value of the rate of decay of the initial and all subsequent states (intensity?) of compound fractional Poisson process (1), (3) is equal

$$\Lambda^{1/\nu} = 2.2129 \; [day^{-1}],$$

the average exponent of the fractional derivative – the hereditarily parameter is equal

$$\nu = 0.7849,$$

the process stability parameter (parameter b is taken from the Table 1) takes the value

$$(2b+1)\nu \approx 1,8675.$$

The main result: the seismic process has memory properties, and random events cannot be considered independent. The statistical dependence of random events determines the determinism of the process, a measure of this can be the value of $1-\nu$.

Conclusions

1. The considered model has three special properties: complexity, power-law scaling, and hereditarity. Each of these properties contributes to the statistical features of the model.

2. Complexity (compositeness) can take different forms, but a special case gives power-law scaling, which causes instability and non-stationarity of the process. In seismology, these are foreshocks, mainshock and aftershocks. In laser physics, this is a gigantic impulse. In condensed matter physics, this is explosive boiling. In the infectious sciences, this is an epidemic. In investments, this is a financial disaster. Scaling through scale consolidation generates divergences in the statistical characteristics of the process.

3. But the main feature of the model is its hereditarity, which generates the consolidation of random events in time with an infinite correlation radius, according to distribution (5). The process slows down, and due to this, the collectivization of events occurs.

4. As has been shown, scaling and hereditarity produce a multiplier effect of the consolidation of events, which results in a catastrophic rather than a gigantic phenomenon. The second term of the process dispersion is responsible for the coherence.

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Список литературы

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Thank you for attention.